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Thermofractals, Non-Additive Entropy, and *q*-Calculus

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Abstract: Non-additive entropy is obtained through the thermodynamic description of a system with a fractal structure in its energy-momentum space, called a thermofractal. The entropic parameter, q, is determined in terms of the fractal structure parameters. The characteristics of the thermofractals are determined by two parameters associated with the number of degrees of freedom of the fractal structure and the scale. The parameter q, of non-extensive thermodynamics, has a physical meaning related to the number of degrees of freedom of the thermofractal. The two types of thermofractals are distinguished by the value of $q > 1$ or $q < 1$. Studying the group of transformations of the fractal system, we identify three different classes of transformations and their mathematical expressions. For one class of transformations of thermofractals, the group is isomorphic with q-calculus. Another class of transformations led to new mathematical expressions that extended the deformed q-algebra. Finally, we comment regarding the applications of the results obtained here for different areas such as QCD and scale-free networks.

Keywords: non-additive entropy; non-extensive thermodynamics; fractals; thermofractals; q-calculus; q-algebra; deformed algebra; Lie groups

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Citation: Deppman, A. Thermofractals, Non-Additive Entropy, and *q*-Calculus. *Physics* **2021**, *3*, 290–301. https://doi.org/10.3390/physics3020021

Received: 15 March 2021
Accepted: 28 April 2021
Published: 4 May 2021

1. Introduction

In the last quarter of the twentieth century, two new concepts changed the landscape of science. The fractal geometry introduced by B. B. Mandelbrot [1] generalized the Euclidean geometry and included many natural geometric shapes in a rigorous mathematical formalism developped, among others, by F. Haussforff [2]. The non-extensive statistics introduced by C. Tsallis [3] generalized the Boltzmann–Gibbs statistics and included several dynamical systems in the formalism of thermodynamics.

The self-similarity, i.e., the fact that the system is similar to its constituents when appropriately scaled, is a prominent feature of fractals. Two properties of fractals are determinant in shaping this characteristic—namely, a complex internal structure and the scaling invariance of the system [1]. They are present in physical, biological, social, and computational systems [4–14].

Non-extensive thermodynamics result primarily from the new entropy formula, given by [3,15,16]

$$S_q = -k\sum_i p_i^q \ln_q p_i = k\frac{1-\sum_i p_i^q}{q-1}\,, \tag{1}$$

where p_i is the probability of the ith event, and q is a parameter called the entropic index. The sum includes all possible events, and, in the present case, it includes all possible single-particle states. This definition of entropy implies that

$$\frac{S_q(A+B)}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1-q)\frac{S_q(A)}{k}\frac{S_q(B)}{k}\,, \tag{2}$$

where S_A, S_B, and S_{A+B} are the entropies of the independent systems A and B and the combined system $A+B$, respectively. Equation (2) reflects the non-additivity of the entropy. The parameter q provides a measure of the non-additivity of the entropy or of the non-extensivity of the associated thermodynamics [15,16]. The non-additivity of the entropy introduces some formal difficulties in dealing with the Tsallis statistics, and a deformed algebra, called q-calculus, was introduced [17] to overcome those difficulties.

The concepts of fractals and generalized entropy are not completely unrelated, and the Tsallis statistics, S_q, were proposed to include fractal distributions in the realms of statistical mechanics [3]. The relations between fractals and non-extensivity can be addressed within a new perspective with the concept of thermofractals [18]. Thermofractals were introduced in the context of high energy collisions, where S_q gives an accurate description of the experimental distributions of the multiparticle production process. However, the applicability of these systems goes beyond quantum chromodynamics (QCD) and can be used with any system that is described in terms of the Yang–Mills field theory [4]. One of the main aspects of thermofractals is the fact that the number of degrees of freedom is independent of the system size. This is the reason why non-extensivity emerges in those systems.

The present work provides a brief review on thermofractal systems, which were previously discussed in detail in [19–21]. Then, for the first time, we proceed to investigate the transformation properties of thermofractals. We show that these transformations follow a Lie algebra similar to q-calculus.

2. Thermofractals

There are two types of thermofractals [19], and each one presents the following properties:

(I) It is a thermodynamic system equipped with a complex structure with N components that are, themselves, thermofractals with the same properties, forming a hierarchic structure. The system has a total energy, $U = E + K$, where E is the total internal energy of its components and K is the total kinetic energy of those components.

(II) In the case of the thermofractal type-I, the internal energy E and the kinetic energy K of the system are such that the ratio $\chi = E/K$ follows a distribution $\tilde{p}(\chi)$. In the case of the thermofractal type-II, it is the ratio $\chi = E/U$ which follows a distribution $\tilde{p}(\chi)$.

(III) At some level of the internal structure, the fluctuations of the internal energy of the components of the system are small enough to be disregarded and their internal energy can be considered constant.

For practical reasons, it is convenient to write the scale-free variable as

$$\chi = \frac{\varepsilon}{\Lambda}, \tag{3}$$

where ε refers to the energy of the thermofractal and Λ is a reference energy, which is used as a scale parameter. In the following, both forms are used to refer to the scale-free variable.

As discussed below, the systems that present these properties must satisfy a recursive equation for the probability density, $\tilde{p}_{\pm}(\varepsilon)$, for which the only solution is:

$$\tilde{p}_{\pm}(\chi) = \left[1 \pm (q-1)\frac{\varepsilon}{\lambda}\right]^{\frac{\mp 1}{q-1}} \tag{4}$$

with $\lambda = (q-1)\Lambda$ and $q > 1$ being related to N, as shown below. The signs in the argument and in the exponent are to be taken in pairs $(+,-)$ or $(-,+)$, resulting in only two possible combinations. λ is a reference energy, which can be understood as the typical energy per degree of freedom of the system. In Equation (4), there are two instances of the q-experimental functions, since $\varepsilon \geq 0$ without (sign '$+$' in the argument) and with (sign '$-$' in the argument) a cut-off. These functions are related to the Tsallis statistics, and here they represent the connection between the thermofractals and the non-extensive statistics.

Examples of systems that follow thermofractal dynamics are the Yang–Mills fields [4] and the non-extensive self-consistent hadron [18]. In the case of Yang–Mills fields, the two fundamental features of a fractal are determined by the renormalization group equation, which describes the scaling properties of the system, and by the fact that the effective parton, introduced with the one-particle irreducible description of the fields, has an internal structure associated with the vacuum polarization effects. QCD, as a particular instance of Yang–Mills field theory, can be understood in terms of the thermofractal structure with the value of q resulting in a good agreement with the data [4]. The second example is related to the Hagedorn's self-consistent thermodynamics [22] and with the Chew–Frautschi bootstrap model of hadrons [23], where the self-similarity of the system is a fundamental property. The use of thermofractals to describe hadronic structures or neutron-star structures can be found in Refs. [24,25].

A fractal structure, similar to that observed in thermofractals, can be found also in non-thermodynamic systems such as the spreading dynamics of viruses transmitted by close contact. A description of the SARS-COV-2 virus, which takes into account the observed fractal patterns, gives expressions similar to those obtained for thermofractals [26]; however, the signs in the argument and in the exponent are opposite. The results are positive q-exponential functions, which are not normalized and cannot be understood as probability densities, but rather describe the explosive epidemic process. This is discussed below regarding the possible relations between the fractal structure in thermofractals and in scale-free networks.

To prove Equation (4), let us recall the thermodynamic potential,

$$\log \Xi = \int_{-\infty}^{\infty} d^{3N}p \exp\{-\beta U\}\,, \tag{5}$$

where p is the momentum of the components of a system with total energy U. Consider that

$$K = \frac{p^2}{2m}\,, \tag{6}$$

with $p^2 = \sum_i p_i^2$, where the index i refers to all components of the momenta of all particles in the system. Then, the topological term, arising as one integrates over the energy instead of the momentum [20,21],

$$\int_{-\infty}^{\infty} d^{3N}p \rightarrow \int_0^{\infty} dK\, S_D K^{3N/2-1}[\tilde{p}(E)]^{\nu} dE\,, \tag{7}$$

where S_D is the surface term being independent of K, transforms to:

$$\log \Xi \propto \int_0^{\infty} dK\, S_D K^{3N/2-1} \exp\{-\beta U\}[\tilde{p}(E)]^{\nu} dE\,. \tag{8}$$

In the equations above, ν and N are parameters that completely define the thermofractal structure. The meaning of the parameter ν is discussed below, after Equation (17). It is shown that it is more convenient to describe the fractal structure in terms of another pair of parameters, q and λ, which can be expressed in terms of N and ν.

Due to the property (II) for thermofractal type-I, one can write $U = \alpha_+ K$ with

$$\alpha_+ = 1 + \frac{E}{K}\,. \tag{9}$$

The same property will give, for thermofractal type-II, the relation $K = \alpha_- U$ with

$$\alpha_- = 1 - \frac{E}{U}\,. \tag{10}$$

Let us define the probability density

$$\begin{aligned} p_{\pm}(U) = \int_0^{\infty} \frac{S_D K^{3N/2-1} \exp\{-\alpha_{\pm}^{\pm 1}\beta K\}[\tilde{p}_{\pm}(E)]^{\nu} dK}{\log \Xi} = \\ A \int_0^{\infty} d(\beta K)\, (\beta K)^{3N/2-1} \exp\{-\alpha \beta K\}[\tilde{p}_{\pm}(E)]^{\nu}\,, \end{aligned} \tag{11}$$

where A is a normalization constant. Integrating over βK results in

$$p_{\pm}(U) = A_{\lambda}\, \alpha_{\pm}^{\mp 3N/2}[\tilde{p}_{\pm}(E)]^{\nu}\,. \tag{12}$$

Considering the property (II) and the self-similarity of the fractal structure, Equation (12) becomes a recurrence formula with the energy distribution of the thermofractal being determined in terms of the energy distributions of its components. Recurrence formulas are usually found in fractals. They can be written in the form of coupled-equations, since the main system is a thermofractal similar to its components, then their probability densities must be identical when expressed in a scale-free form. To include the scaling property, one writes the probability densities in terms of the scale-free variable, χ, and Equation (12) becomes:

$$p_{\pm}(\chi) = A_{\lambda}[1 \pm \chi]^{\mp 3N/2}[\tilde{p}_{\pm}(\chi)]^{\nu}\,. \tag{13}$$

The index λ is added to make clear that the normalization constant is written in terms of the scaling parameter. Due to the self-similarity of the thermofractal, the two probabilities must be the same, that is,

$$p_{\pm}(\chi) = \tilde{p}_{\pm}(\chi)\,. \tag{14}$$

The use of Equation (6) in the approach presented here may give the erroneous impression that thermofractals are essentially non-relativistic systems. However, this is not true, and, as it is shown in [4], a relativistic system presents a similar fractal structure. In fact, the main ingredients to obtain the fractal structure are the scale-invariance and a complex internal structure.

2.1. Recurrence Formulas and Tsallis Distribution

Observe that Equations (13) and (14) form a pair of coupled equations that must be solved simultaneously. In those formulas one has, at the left-hand side, the probability density of the system, $p_{\pm}(\chi)$, and, on the right-side, the probability density of its constituents, $\tilde{p}_{\pm}(\chi)$. In both cases, they are written in terms of the scale-invariant term χ.

As the thermofractals are constituted by thermofractals and since they are equal when the differences in scale are removed, the two probability densities must be equal because they represent the same properties of identical objects. For the sake of clarity and simplicity, we used $\mu = 3N/2$ in the following calculations. μ is the number of degrees of freedom (n.d.f.) of the N particles system if one disregards its internal structure, i.e., in the limit, it can be considered as an ideal gas.

The only solutions satisfying those equations have the general form:

$$\tilde{p}_{\pm}(\chi) \propto [1 \pm \chi]^{\alpha} \tag{15}$$

with α being a constant that to be determined. Substituting this expression in Equation (13), one obtains:

$$A_{\lambda}[1 \pm \chi]^{\alpha} = A_{\lambda}[1 \pm \chi]^{\nu\alpha \mp \mu}\,. \tag{16}$$

Hereby, the constant α must be such that $\alpha = \nu\alpha \mp \mu$, and thus,

$$\alpha = \mp \frac{\mu}{1-\nu}\,. \tag{17}$$

The equation above shows that the parameter α is proportional to μ; therefore, α will be called n.d.f. of the thermofractal.

The denominator in Equation (17), $1-\nu$, gives the fraction of the n.d.f. contained in the N components in the thermofractal, and ν, therefore, is the fraction of the n.d.f. beyond those of the N components, since the property (II) also implies that the components have an internal structure with the same number of components. The fact that ν is different from zero means that, when transferring some energy to the thermofractal, this energy will be distributed to its N components in such a way that it goes only partially to their kinetic energy, and the remainder is transferred to their internal components. The larger the fraction ν the deeper the energy is distributed in the fractal structure.

To obtain the q-exponential expression in the forms discussed in Section 3, the parameter q is introduced, such that

$$\frac{1}{q-1} = \frac{\mu}{1-\nu}\,, \tag{18}$$

and the reduced scale parameter,

$$\Lambda = \frac{\lambda}{q-1}\,. \tag{19}$$

Equation (18) shows that $(q-1)^{-1} = \alpha$ is the n.d.f. of the thermofractal, and λ is the reference energy *per degree of freedom*, and works as a scaling parameter of the fractal. Substituting the expressions above in Equation (15), one gets:

$$\tilde{p}_{\pm}(\varepsilon) = \left[1 \pm (q-1)\frac{\varepsilon}{\lambda}\right]^{\frac{\mp 1}{q-1}}\,, \tag{20}$$

which is exactly the result we wanted to prove (see Equation (4)).

The parameters q and λ fully characterize the thermofractal and can be expressed in terms of N and ν. Equation (4) is a q-exponential function of ε only in the case where the scaling symmetry is broken. This will always happen because of the property (III) but can also occur for different reasons depending on the system considered.

From Equations (18) and (19), $q \geq 1$. This is, indeed, a consequence of dealing with a thermodynamic system, where conservation laws apply, and the q-exponential function describes a probability density that has to be normalizable. For non-thermodynamic systems, such as the virus spreading dynamics [26], these constraints may not be valid.

It is interesting to analyze the case $q \to 1$ in more detail. Observe that, from Equation (18), the case $q = 1$ corresponds to $\nu = 1$. This means that the n.d.f. carried by the N particles of the system, disregarding their internal structure, gives a inifnitesimal contribution to the total n.d.f. of the thermofractal. Due to property (III), the total n.d.f. corresponds to those of the components at the lowest hierarchic level of the structure, and, in this case, the system corresponds to an ideal gas that can be described by the Boltzmann–Gibbs statistics. Thus, the correspndence between the thermofractal and an ideal gas in the limit $q \to 1$ is identical to the correspondence between the Tsallis statistics and the Boltzmann–Gibbs statistics in the same limit.

2.2. Thermofractal Transformation Group

In this section, we describe three classes of transformations for thermofractals. The first is the self-similar transformation, the second is the scale transformation, and the third is the complexity transformation.

The first and most obvious transformation is the self-similar transformation. In this case, the energy of the system, ε, and the scale parameter, λ, are transformed in such a way that the ratio $\xi = \varepsilon/\lambda$ remains unchanged. As one can observe from the probability distributions in Equation (4), all the characteristics of the system remain exactly the same; therefore, this transformation is equivalent to the identity transformation as an expression of the self-similarity of the fractals. We adopt the operator symbols, T_a such that

$$T_a e_q(\varepsilon/\lambda) = e_q(a\varepsilon/a\lambda)\,, \tag{21}$$

Clearly, for any real $a \neq 0$, $T_a = I$, where I is the identity operator.

The second transformation is related to the scale transformation of the thermofractal, as given by the variable ξ. This can be accomplished by changing λ, that is the scale parameter, for a fixed value of ε. As this is equivalent to a fixed λ and varying ε, we keep this latter alternative, for simplicity. In these conditions, the rate of variation of $\tilde{p}_q(\varepsilon)$ due to changes in ε is given by

$$\frac{\partial \tilde{p}_\alpha(\varepsilon)}{\partial \varepsilon} = \tilde{p}_\alpha^q(\varepsilon)\,, \tag{22}$$

where the index α is used instead of q to indicate the n.d.f. as soon as they are related by Equation (17). Thereby, when the infinitesimal scale transformation, $\varepsilon \to \varepsilon' = \varepsilon + d\varepsilon$, is performed, the probability density changes by

$$\tilde{p}_\alpha(\varepsilon) \to \tilde{p}_\alpha(\varepsilon + \delta\varepsilon) = \left[1 \pm \frac{1/\lambda}{1+(1-q)\varepsilon/\lambda}\delta\varepsilon\right]\tilde{p}_\alpha(\varepsilon)\,. \tag{23}$$

In the limit $\varepsilon/\lambda \gg 1/(q-1)$, the result above can be simplified to

$$\tilde{p}_\alpha(\varepsilon + \delta\varepsilon) = [1 \pm N\,d\log\varepsilon]\tilde{p}_\alpha(\varepsilon)\,. \tag{24}$$

In Equations (23) and (24), the terms between brackets are the infinitesimal scaling operator. The operator S_δ represents the scale transformation of the thermofractal, such that

$$S_\delta\,\tilde{p}_\alpha(\varepsilon) = \tilde{p}_\alpha(\varepsilon + \delta)\,, \tag{25}$$

for an infinitesimal real δ. From Equation (23), one gets:

$$S_\delta = \left[1 \pm \frac{\delta/\lambda}{1+(1-q)\varepsilon/\lambda}\right]. \tag{26}$$

The following properties can be easily verified:

$$\begin{cases} S_\mu S_\nu = S_{\mu+\nu}\,, \\ S_0 = I\,. \end{cases} \tag{27}$$

This transformation is directly related to the symmetry break, since it modifies only ε, while λ remains constant.

The third transformation of thermofractals is the complexity transformation, where the exponent $\alpha = 1/(q-1)$ is transformed. The fact that the n.d.f. is independent of the size of the system, as discussed in the last section, allows two classes of transformations of a thermofractal. These transformations are those that modify, independently, each of the characteristic parameters, q and λ.

For simplicity, this transformation will be studied in terms of α instead of q; however, it is straightforward to extend the derivation to variations in the parameter q. Changing α leads to a rate of variation of $\tilde{p}_\alpha(\varepsilon)$ as given by

$$\frac{d\tilde{p}_\alpha(\varepsilon)}{d\alpha} = \log(1+\varepsilon)\,d\alpha\,\tilde{p}_\alpha(\varepsilon)\,; \tag{28}$$

therefore,

$$\tilde{p}_{\alpha+d\alpha}(\varepsilon) = [1 + \log(1+\varepsilon)\,d\alpha]\tilde{p}_\alpha(\varepsilon)\,. \tag{29}$$

When performing this transformation, one must keep in mind that the relation $\alpha\lambda = \Lambda$, with Λ being constant, remains valid when α varies. This happens because for this transformation, the size of the system, ε/Λ, does not change.

The operator Q_δ represents the complexity transformation of the thermofractal, that is,

$$Q_\delta\,\tilde{p}_\alpha(\varepsilon) = \tilde{p}_{\alpha+\delta}(\varepsilon)\,, \tag{30}$$

and from Equation (29), one obtains

$$Q_\delta = [1 + \log(1+\varepsilon)\,\delta]\,. \tag{31}$$

The following properties can be easily verified,

$$\begin{cases} Q_\mu Q_\nu = Q_{\mu+\nu}\,, \\ Q_0 = I \end{cases} \tag{32}$$

for μ and ν infinitesimals. This transformation represents an increase in the complexity of the thermofractal, as this increases the n.d.f. of the system. It is related to the modification of the structure of the thermofractal associated with the number of internal parameters, N, or the fraction ν of the degrees of freedom of the internal components.

Finite transformations can be easily obtained by successive applications of the operators. One obtaines, for the scale transformation, the integral operator:

$$S_a = \lim_{\delta\to\infty}[S_\delta]^{a/\delta} = 1 \pm \int_0^a \frac{d\varepsilon/\lambda}{1+(1-q)\varepsilon/\lambda}\,. \tag{33}$$

In the case of the complexity transformation, one has:

$$Q_a = \lim_{\delta\to\infty}[Q_\delta]^{a/\delta} = 1 + \log(1+\varepsilon)\int_0^a d\alpha\,. \tag{34}$$

In Section 4, the use of these operators in the context of the q-calculus is discussed.

3. Group of Transformations of the q-Exponential Function

The q-exponential function is frequently found in the non-extensive systems that are described in terms of the Tsallis statistics. It represents the inverse function of the q-logarithm function associated with the entropic formula in Equation (1). Here, the group of transformations of the q-exponential is studied, identifying two classes of transformations, the *reflection transformation in the domain space*, where $x \to y = -x$, and the *q-reflection in the parametric space*, where $q \to q' = 2-q$.

The q-exponential function is defined [3] by:

$$e_q(x) = \begin{cases} [1+(1-q)x]^{\frac{1}{1-q}}\,, x \geq \frac{-1}{1-q}\,, \\ 0\,;\, x < \frac{-1}{1-q}\,, q < 1\,, \end{cases} \tag{35}$$

and the second equality is known as the Tsallis cut-off. The function is characterized by its real argument, x, and the parameter q. These two parameters determine two classes of functions corresponding to the values $q < 1$ and $q > 1$. The exponential function is one particular function pertaining to both classes, with $q \to 1$, when the statistics become additive. We studied the possible transformations in the domain space and in the parametric space that transform the objects in the class of q-exponential into themselves.

In high-energy physics (HEP), more frequently a form similar to that of Equation (35) in all respects but with the value of $q > 1$ is used [27–35]. This form can be obtained from Equation (35) by considering the case where $q > 1$ [36,37], which one conveniently writes as:

$$e_q(y) = \begin{cases} [1-(q-1)y]^{\frac{-1}{q-1}}\,, y \leq \frac{-1}{q-1}\,, \\ 0\,;\, y > \frac{1}{q-1}\,, q > 1\,. \end{cases} \tag{36}$$

Here the symbol in the argument is changed to y for the sake of clarity. In this case, the cut-off has an inferior limit, in contrast to the case defined in Equation (35). This aspect suggests the transformation $y \to x = -y$, which results in:

$$e_q(x) = \begin{cases} [1+(q-1)x]^{\frac{-1}{q-1}}\,, x \geq \frac{-1}{1-q}\,, \\ 0\,;\, x < \frac{-1}{1-q}\,, q > 1\,. \end{cases} \tag{37}$$

This is the form of the q-exponential function normally used in HEP.

Equation (36) can be obtained from Equation (35) by applying the q-reflection transformation, when the parameter $q < 1$ is transformed into $q > 1$. Therefore, the q-exponential form used in HEP can be obtained from that defined in the original work [3] by the application of the q-reflection followed by the reflection of the domain space.

The following properties are straightforward: the (a) double application of the q-reflection is the identity transformation; (b) double application of the reflection of the domain space is the identity and the transformations in the parametric and in the domain spaces commute. An obvious consequence is that, if one applies the q-reflection and the domain-reflection in the form given in Equation (37), the form in Equation (35) is obtained, which can be easily checked by the reader.

In all cases above, the limit $q \to 1$ leads to the exponential function, where Equation (35) results in an increasing exponential, while Equation (37) results in a decreasing exponential. For simplicity, we adopted the operator symbols $R_x(\delta)\, e_q(x) = e_q(x+\delta)$ to indicate the domain-space reflection and $R_q(\delta)\, e_q(x) = e_{q+\delta}(x)$. The following properties of the operators are straightforwardly obtained:

$$\begin{cases} R_x^2 = I\,, \\ R_q^2 = I\,, \\ R_x R_q = R_q R_x = I\,. \end{cases} \tag{38}$$

The case $q < 1$ might have interesting consequences in hadronic systems, and could be related to the asymmetry between matter and antimatter in multiparticle production [38].

4. Discussion

In Section 2, the thermofractal system is described, showing that there are two types of thermofractals. Each type has its own recurrence formula, which was obtained from the study of their fractal structure. Section 2.1 shows that the solutions of the recurrence formulas lead to q-exponential functions pertaining to different classes, one with $q > 1$ and the other with $q < 1$. In this way, the reflection transformation of the q-exponential domain space and the q-reflection transformation of the parametric space are associated to concrete aspects of the structure of the fractal systems.

The fact that thermofractals follow q-exponential distributions is associated with the Tsallis statistics [19] and shows how the non-additive statistics emerge, in this case, from the fractal structure of energy momentum space [20] of the system. In this regard, two aspects of the thermofractal structure are relevant, namely, the independence of its number of degrees of freedom from the system size, and the scale-symmetry break. The latter is fundamental to obtain from the solutions of the recurrence formulas with fixed scale-parameter, the q-exponential distribution.

Different mechanisms for the emergence to the non-extensivity have been formulated [39–43], and, in particular, the approach via small systems [40,44,45] has important similarities with the thermofractal. Research showed that finite ideal gas followed q-exponential distributions, which demonstrates that Tsallis statistics have a more fundamental role than Boltzmann–Gibbs statistics [44]. Regarding the size of the system, the value for $q - 1 \propto 1/\alpha$, with α being the number of particles, is invariably obtained. This relation, however, shows that the limit of the Boltzmann-Gibbs statistics, when $q \to 1$, is rapidly achieved. A consequence of this result is that size of the system, alone, cannot explain the ubiquity of systems following Tsallis statistics. With the thermofractal structure, this problem is overcome since the independence of the number of degrees of freedom with size will make the system remain non-extensive even for large systems [21].

In Section 2.2, the thermofractal transformation group was studied, that is, the set of transformations that can be applied to thermofractals while maintaining its structure. Three classes of transformations are described: the self-similar transformation, which represents the scaling invariance of the fractal structure and corresponds to the identity transformation when the size of the system changes in accordance with the scale parameter;

the scale transformation, which changes the size of the thermofractals without changing the number of degrees of freedom; and the complexity transformation, which changes the number of degrees of freedom.

In Section 3, the dual character of the q-exponential function was discussed, $e_q(x)$, with $q < 1$ or $q > 1$, and we studied the group of transformations for the family of q-exponential functions. We was observed that there is a group associated to the parametric space, and introduced the *q-reflection transformation*, which transforms $q \to q' = 2 - q$. We also identified the *reflection transformation* of the domain space, which transforms the argument of the function as $x \to y = -x$. We showed that the two transformations are isomorphic. With these two transformations, the different forms of the q-exponential can be obtained. The transformations of the q-exponential class of functions can be written in terms of the thermofractal transformations since

$$\begin{cases} S_{-2\varepsilon} = R_\varepsilon \,, \\ Q_{2(1-q)} = R_q \,. \end{cases} \tag{39}$$

Due to Equation (38), $S_{-2\varepsilon}Q_{2(1-q)} = Q_{2(1-q)}S_{-2\varepsilon} = I$. Equation (39) result leads to the non-trivial conclusion that the thermofractal of one type can be converted into the other type by a q-reflection transformation.

The results above show the relations between the thermofractal transformation group and the q-exponential transformations, which are associated with the q-calculus. Only tranformations associated with the q-reflection and the domain-reflection performed simultaneously can be associated with the transformations of the thermofractal group. However, it is possible to conjecture that the q-reflection can be a transformation of a system more general than the thermofractals, where non-normalizable distributions would be allowed.

For instance, the fractal aspects of the virus epidemic dynamics, described in [26], bear many similarities with the thermofractal structure but lead to non-normalizable distributions that are useful to describe the explosive epidemic process that happens under uncontrolled virus spreading. The q-reflection, which is a dual transformation of the domain-reflection, as verified in this paper, corresponds to a change of a sign in the argument of the q-exponential function. In the thermofractal structure, this would correspond to the change of the relation between Λ and λ in Equation (19) to

$$\Lambda = -\frac{\lambda}{q-1} \,. \tag{40}$$

It is easy to see that this transformation results in diverging q-exponential functions similar to those found in the study of the epidemic dynamics [26]. In this regard, an investigation on the relations between scale-free networks and fractal structures in Yang–Mills fields would be an interesting subject.

The self-similar transformation keeps the scale-free variable constant and does not change the probability density of the thermofractal. The scale transformation changes the size of the system, as measured by the variable $\xi = \varepsilon/\lambda$. This transformation follows from the scaling-symmetry break, associated with the property (III). In this case, a particular scale, λ, becomes a preferential scale for the system, and the q-exponential distributions are consequences of that property.

The scale transformation is described by Equations (23) and (24), and the latter is an approximation for large systems. These show how the probability density is modified when the scale of the system, measured in units of the scale parameter λ, changes. The group has an associated Lie algebra that is isomorphic with the q-algebra introduced in the context of the non-additive entropy [17]. This can be proved in a more direct way. An immediate consequence of Equation (33) is that, for a finite change, $\Delta\varepsilon = \eta$, the probability density changes by

$$\tilde{p}_\alpha(\varepsilon/\lambda + \eta/\lambda) = \tilde{p}_\alpha(\varepsilon/\lambda) \pm \int_0^\eta \frac{p_\alpha(\varepsilon/\lambda)d\varepsilon/\lambda}{1+(1-q)\varepsilon/\lambda}\tilde{p}_\alpha(\varepsilon/\lambda) \,, \tag{41}$$

for thermofractals of type-I (sign $+$) or type-II (sign $-$). However, Equation (41) is exactly the integration according to the q-calculus [17], for $q < 1$ and $q > 1$. Thus, the scale transformation of the thermofractals gives a physical significance for the deformed q-calculus developed in the context of non-additive entropy.

The complexity transformation is expressed in terms of Equation (29). This shows how the probability density changes when the number of degrees of freedom of the thermofractal changes. As the number of degrees of freedom increases, the q-exponential function approaches the exponential. In the limit of n.d.f. going to infinity, the Boltzmann–Gibbs statistic becomes appropriate to describe the thermofractal system. This transformation can be associated with a new operation in the deformed q-calculus, which, to the best of our knowledge, has not been established yet. From Equation (34), we obtain

$$\tilde{p}_{\alpha+d\alpha}(\varepsilon) = \tilde{p}_{\alpha}(\varepsilon) + \log(1+\varepsilon)\tilde{p}_{\alpha}(\varepsilon)d\alpha\,, \tag{42}$$

and, for finite values, ω,

$$\tilde{p}_{\alpha+\omega}(\varepsilon) = \tilde{p}_{\alpha}(\varepsilon) + \log(1+\varepsilon)\int_0^{\omega} \tilde{p}_{\alpha}(\varepsilon)d\alpha\,. \tag{43}$$

This transformation and the related q-calculus operation can be useful for systems where the complexity increases in a way so that two different systems are combined forming a single one. This might be of a particular interest in the study of small systems where the complexity increases with the number of particles.

The relations between the q-calculus, fractional derivatives, and Haussdorff derivatives [46] are an interesting subject that may have connections with the approach adopted here. The same fractal structures studied here were obtained in a slightly different way in the study of Yang–Mills fields [4] and in locally transmitted information such as the dynamics of epidemic spreading [26].

5. Conclusions

In this paper, the thermofractal structure is studied identifying the two types of the system, which differ by the scaling properties of the fractal structure. The connections between the two types of thermofractals to the two classes of q-exponential, with $q < 1$ and $q > 1$, is established. The thermofractal transformation group was analyzed obtaining the one-parameter infinitesimal generators of the Lie group.

The existence of an isomorphism between the Lie-algebra of the thermofractal group and the q-algebra of the Tsallis statistics is demonstrated. From the group of transformations of the thermofractals, the q-calculus is derived demonstrating the connections of the deformed algebra with the physical structure of the system. The q-algebra is, thus, obtained from the thermofractal transformation properties.

It is shown that one of the thermofractal transformation subgroups, the complexity transformation, suggests new operations for q-calculus and q-algebra.

Funding: This research was funded by Conselho Nacional de Desenvolvimento Científico e Tecnológico grant number 304244/2018-0, and Fundação de Amparo à Pesquisa do Estado de São Paulo under grant 2016/17612-7 and Project INCT-FNA Proc. No. 464 898/2014-5.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Acknowledgments: The author thanks the suggestions given by C. Tsallis and E. Borges, who brought to his attention the work in Ref. [46]. This work is partially supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq-Brazil) and by Project INCT-FNA Proc. No. 464 898/2014-5 and by FAPESP under grant 2016/17612-7.

Conflicts of Interest: The authors declare no conflict of interest.

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